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TRANSFORMATION OF SERIES BY MEANS OF FUNCTIONS ADMITTING A RECURRENT RELATION.

By W. C. Brenke.

1. By transformation of trigonometric series certain results have been obtained as to the behavior of these series.* We wish to call attention to a generalization contained in the following theorem.

Theorem 1. *Let the function* ψ_n *satisfy a recurrent relation of the form*

$$\varphi_n \psi_n = \alpha_n \psi_{n+r} + \beta_n \psi_{n+s}, \qquad (n = 1, 2, 3, \cdots), \tag{1}$$

where all the expressions involved are defined for each value of n, *and* φ_n *does not vanish identically for any value of* n. *They may be constants or functions of a variable,* x. *The numbers* r *and* s, *which are independent of* n, *are not restricted to integral values, but* $r - s$ *is a positive integer.*

Then the series $\sum_1^\infty u_n \psi_n$, *where* u_n *is defined for each value of* n, *converges provided that*

(α) *the series* $\sum_{n=1}^\infty \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r}$ *converges, and that*

(β) *either* $\lim_{n=\infty} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} = 0$, *or that* $\lim_{n=\infty} \frac{\alpha_n}{\varphi_n} u_n \psi_{n+r} = 0$.

We then have

$$\sum_1^\infty u_n \psi_n = \sum_{n=1}^{r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} + \sum_{n=1}^\infty \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r}. \tag{2}$$

To obtain this result we replace ψ_n in the polynomial $\sum_1^k u_n \psi_n$ by its value from (1), so that

$$\sum_{n=1}^k u_n \psi_n = \sum_{n=1}^k \left(\frac{\alpha_n}{\varphi_n} \psi_{n+r} + \frac{\beta_n}{\varphi_n} \psi_{n+s} \right) u_n = \sum_{n=1}^k \frac{\alpha_n}{\varphi_n} u_n \psi_{n+r} + \sum_{n=1-r+s}^{k-r+s} \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \psi_{n+r};$$

hence,

$$\sum_{n=1}^k u_n \psi_n = \sum_{n=1}^{r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s} + \sum_{n=1}^k \left(\frac{\alpha_n}{\varphi_n} u_n + \frac{\beta_{n+r-s}}{\varphi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r} - \sum_{n=k+1}^{k+r-s} \frac{\beta_n}{\varphi_n} u_n \psi_{n+s}; \tag{3}$$

if we allow k to increase indefinitely, the theorem follows.

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* Schlömilch: Comp. d. höh. Analysis, vol. 1, § 40; Lerch: Annales de l'Ecole normale, ser. 3, vol. 12 (1895), p. 351; the writer: Annals of Mathematics, ser. 2, vol. 8 (1907), p. 87.

When the given series $\sum_1^\infty u_n\psi_n$ is known to converge, (β) is a necessary and sufficient condition that the transformed series (α) shall converge and to the value as given by (2).

DEFINITION. We shall refer to the results expressed in (2) as "a transformation of the series $\sum_1^\infty u_n\psi_n$ by the factor φ_n." If the series in the right-hand member of (2) be similarly treated and the process repeated p times, we define the final result as "a transformation of the original series by the factor φ_n^p."

APPLICATION. In practical applications equation (2) furnishes a convergence test for the series on the left, and also facilitates its approximate numerical evaluation, since it is often possible to transform a given series into another which converges more rapidly. Some series can be summed in closed form by this process.

We shall consider cases in which ψ_n is one of the functions $\cos nx$, the nth polynomial of Hermite* $U_n(x)$, the nth polynomial of Legendre $X_n(x)$, or the nth Bessel function $J_n(x)$. In place of (1) we take respectively,

$$(a)\qquad 2\cos\tfrac{1}{2}hx\cos nx = \cos(n+\tfrac{1}{2}h)x + \cos(n-\tfrac{1}{2}h)x;$$

$$(b)\qquad -U_n(x) = \frac{1}{2n+2}U_{n+2}(x) + \frac{2x}{2n+2}U_{n+1}(x);$$

$$(c)\qquad xX_n(x) = \frac{n+1}{2n+1}X_{n+1}(x) + \frac{n}{2n+1}X_{n-1}(x);$$

$$(d)\qquad \frac{2}{x}J_n(x) = \frac{1}{n}J_{n+1}(x) + \frac{1}{n}J_{n-1}(x).$$

Then from (2) we have

$$(a_1)\quad \sum_1^\infty u_n\cos nx = \frac{1}{2\cos\frac{1}{2}hx}\sum_{n=1}^{h}u_n\cos(n-\tfrac{1}{2}h)x + \frac{1}{2\cos\frac{1}{2}hx}\sum_{n=1}^{\infty}(u_n+u_{n+h})\cos(n+\tfrac{1}{2}h)x,$$

provided that $\lim\limits_{n=\infty} u_n = 0$;

$$(b_1)\quad \sum_1^\infty u_nU_n(x) = -\tfrac{1}{2}xu_1U_2(x) - \sum_{n=1}^{\infty}\left(\frac{x}{n+2}u_{n+1} + \frac{1}{2n+2}u_n\right)U_{n+2}(x),$$

provided that $\lim\limits_{n=\infty}\dfrac{u_n}{2n+2}U_{n+2}(x) = 0$;

$$(c_1)\quad \sum_1^\infty u_nX_n(x) = \frac{u_1}{3x} + \frac{2u_2}{5x}X_1(x) + \frac{1}{x}\sum_1^\infty\left(\frac{n+1}{2n+1}u_n + \frac{n+2}{2n+5}u_{n+2}\right)X_{n+1}(x),$$

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* For an interesting application of these polynomials see "Integral-gleichungen, etc.," by Myller-Lebedeff; Math. Ann., vol. 64 (1907), p 390.

provided that $\lim\limits_{n=\infty} u_n X_{n+1}(x) = 0$;

$$(d_1) \quad \sum_1^\infty u_n J_n(x) = \frac{x}{2}[u_1 J_0(x) + \tfrac{1}{2}u_2 J_1(x)] + \frac{x}{2}\sum_1^\infty \left(\frac{u_n}{n} + \frac{u_{n+2}}{n+2}\right) J_{n+1}(x),$$

provided that

$$\lim_{n=\infty} \frac{u_n}{n} J_{n+1}(x) = 0;$$

provided further that either one of the two infinite series in each equation converges.

Formulas (b_1) and (d_1) in general increase the convergence of the given series by replacing u_n by a quantity of the order of magnitude of $u_n \div n$. Formulas (a_1) and (c_1) are useful in increasing the convergence of slowly convergent alternating series. As examples we have

$$(a_2) \quad \sum_1^\infty \frac{(-1)^{n+1}}{n} \cos nx = \frac{1}{2} + \frac{1}{2\cos\frac{1}{2}x} \sum_1^\infty \frac{(-1)^{n+1}}{n(n+1)} \cos(n + \tfrac{1}{2})x;$$

$$(b_2) \quad \sum_1^\infty \frac{1}{n!} U_n(x) = -\frac{x}{2} U_2(x) - \sum_1^\infty \frac{2x+n+2}{2(n+2)!} U_{n+2}(x);$$

$$(c_2) \quad \sum_1^\infty (-1)^v X_n(x) = \frac{1}{3x} + \frac{2}{5x} X_1(x) + \frac{1}{x}\sum_1^\infty (-1)^v \frac{2n+3}{(2n+1)(2n+5)} X_{n+1}(x),$$

where $v = \frac{1}{2}n + 1$ when n is even; $v = \frac{1}{2}(n-1)$ when n is odd;

$$(d_2) \quad \sum_1^\infty (-1)^v n J_n(x) = \frac{x}{2}[J_0(x) + J_1(x)], \ v \text{ as in } (c_2).$$

In connection with the last example it may be remarked that every series of the form

$$\sum_{n=1}^\infty n^p J_n(x),$$

where p is a positive integer, may be summed by reduction by successive steps to the series $\sum\limits_1^\infty J_n(x)$, whose sum is given in § 2 of this paper. It follows that the above series converges for all values of p and of x.

As special cases we have from (a_1) when $x = 0$ and from (c_1) when $x = 1$ respectively,

$$(a_3) \quad \sum_{n=1}^\infty u_n = \frac{1}{2}\sum_1^h u_n + \frac{1}{2}\sum_1^\infty (u_n + u_{n+h});$$

$$(c_3) \quad \sum_1^\infty u_n = \tfrac{1}{3}u_1 + \tfrac{2}{5}u_2 + \sum_1^\infty \left(\frac{n+1}{2n+1} u_n + \frac{n+2}{2n+5} u_{n+2}\right);$$

provided that $\lim\limits_{n=\infty} u_n = 0$, and that one of the two series in each equation converges.

Equation (a_3) is the limit of a special case under (3), namely when, for all values of n, $\psi_n = 1$, $\varphi_n = 2$, $\alpha_n = \beta_n = 1$, $r - s = h$. If with these special values we apply (3) k times in succession and if we put, symbolically,

$$\begin{aligned}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^m \equiv\ & (u_1 + u_2 + \cdots + u_h)\\ &+ {}_mC_1(u_{1+h} + \cdots + u_{2h}) + {}_mC_2(u_{1+2h} + \cdots + u_{3h})\\ &+ \cdots + {}_mC_m(u_{1+mh} + \cdots + u_{h+mh}),\end{aligned}$$

we find

$$\begin{aligned}s_{kh} = u_1 + u_2 + \cdots + u_{kh} &= \sum_{n=1}^{k} \frac{1}{2^n}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^{n-1}\\ &+ \frac{1}{2^k}[s_{kh} + {}_kC_1(s_{kh} - s_h) + {}_kC_2(s_{kh} - s_{2h}) + \cdots + {}_kC_{k-1}(s_{kh} - s_{(k-1)h})]\end{aligned} \tag{4}$$

We shall show that the expression in the last line vanishes when $k = \infty$ provided that the series $\sum_1^\infty u_n$ converges.

The expression in question may be written in two parts,

$$\frac{s_{kh} + {}_kC_1(s_{kh} - s_h) + \cdots + {}_kC_r(s_{kh} - s_{rh})}{2^k}$$
$$+ \frac{{}_kC_{r+1}(s_{kh} - s_{(r+1)h}) + \cdots + {}_kC_{k-1}(s_{kh} - s_{(k-1)h})}{2^k}.$$

Since $\sum_1^\infty u_n$ converges, r can be taken so large that for sufficiently large values of k and for all larger values the largest of the quantities in the parentheses in the second fraction shall be less in absolute value than a preassigned positive quantity e. Hence the fraction itself will be less than e.

Also, a positive constant M exists such that the first fraction is less in absolute value than the quantity

$$M\frac{1 + {}_kC_1 + {}_kC_2 + \cdots + {}_kC_r}{2^k}$$

and this vanishes with increasing k, no matter what the value of r.

Hence we obtain the following transformation for any *convergent* series:

$$(a_4)^{*} \qquad \sum_1^\infty u_n = \sum_1^\infty \frac{1}{2^n}(u_1 + u_2 + \cdots + u_h)(1 + u_h)^{n-1},$$

where h is any positive integer, and we have symbolically,

$$(1 + u_h)^m \equiv 1 + {}_mC_1u_h + {}_mC_2u_{2h} + \cdots + u_{mh}; \quad u_nu_m \equiv u_{n+m}.$$

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[*] When $h = 1$ this becomes a transformation given by L. D. Ames: "Evaluation of slowly convergent series," Annals of Mathematics, series 2, vol. 3 (1902), p. 185.

A general type of recurrent relation, of which (1) is a special case when r and s are integers, is contained in the equation,

$$\varphi_n \psi_n = \sum_{i=1}^{h} \alpha_{n,i} \psi_{n+r_i}, \tag{5}$$

h being a fixed positive integer, and $r_1, r_2, \cdots r_h$, any set of integers.

Let p be a fixed positive integer such that $p + r_i > 0$, $i = 1, 2, \cdots, h$. Also let

$$A_{n,i} = \frac{u_n}{\varphi_n} \alpha_{n,i}.$$

Then

$$\sum_{p}^{k} u_n \psi_n = \sum_{n=p}^{k} \sum_{i=1}^{h} A_{n,i} \psi_{n+r_i} = \sum_{i=1}^{h} \sum_{n=p+r_i}^{k+r_i} A_{n-r_i,i} \psi_n$$

$$= \sum_{i=1}^{h} \left[\sum_{n=p}^{k} A_{n-r_i,i} \psi_n - \sum_{n=p}^{p+r_i-1} A_{n-r_i,i} \psi_n + \sum_{n=k+1}^{k+r_i} A_{n-r_i,i} \psi_n \right],$$

where any summation whose upper limit is less than the lower is to be replaced by one in which the upper limit is at least as large as the lower according to the formula

$$\sum_{s}^{s+r-1} = \begin{cases} 0 \text{ if } r = 0, \\ -\sum_{s+r}^{s-1} \text{ if } r < 0. \end{cases}$$

With this convention the above transformation holds for negative and zero values of r_i as well as for positive values.

If now we assume that

$$\lim_{k=\infty} \sum_{i=1}^{h} \sum_{n=k+1}^{k+r_i} A_{n-r_i,i} \psi_n = 0,^{*}$$

we have

$$\sum_{p}^{\infty} u_n \psi_n = -\sum_{i=1}^{h} \sum_{n=p}^{p+r_i-1} A_{n-r_i,i} \psi_n + \sum_{n=p}^{\infty} \left(\sum_{i=1}^{h} A_{n-r_i,i} \right) \psi_n, \tag{6}$$

provided that either one of the two infinite series here involved converges.

As an example we transform the series $\sum_{3}^{\infty} n^2 J_n$ by means of the recurrent relation

$$\left(\frac{2}{x}\right)^2 J_n = \frac{1}{n(n+1)} J_{n+2} + \frac{2}{n^2-1} J_n + \frac{1}{n(n-1)} J_{n-2}$$

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[*] A sufficient condition is $\lim_{n=\infty} A_{n-r_i,i} \psi_n = 0$.

Then

$$\varphi_n = \left(\frac{2}{x}\right)^2;\ h = 3;\ r_1 = 2,\ r_2 = 0,\ r_3 = -2;$$

$$\alpha_{n,1} = \frac{1}{n(n+1)},\quad \alpha_{n,2} = \frac{2}{n^2-1};\quad \alpha_{n,3} = \frac{1}{n(n-1)}.$$

Also p may be any integer greater than 2.

Equation (6) then becomes

$$\sum_3^\infty n^2 J_n = \frac{x^2}{4}\left[\tfrac{3}{2}J_1 + \tfrac{4}{3}J_2 - \tfrac{1}{2}J_3 - \tfrac{2}{3}J_4 + 4\sum_3^\infty J_n\right].$$

As a second simple example we transform the power series $\sum_h^\infty a_n x^n$ by use of the relation

$$(c_0 + c_1 x + \cdots c_{h-1}x^{h-1})x^n = c_0 x^n + c_1 x^{n+1} + \cdots + c_{h-1}x^{n+h-1}.$$

Putting

$$\varphi = c_0 + c_1 x + \cdots + c_{h-1}x^{h-1},$$

equation (6) gives

$$(6') \qquad \varphi\sum_h^\infty a_n x^n = -\sum_{i=1}^{h}\sum_{n=h}^{h+i-2} c_{i-1}a_{n-i+1}x^n + \sum_{n=h}^{\infty}\left(\sum_{i=1}^{h} c_{i-1}a_{n-i+1}\right)x^n,$$

provided that either one of the series converges. Here $c_0, c_1, \cdots, c_{h-1}$ is any arbitrary set of constants, and the general coefficient of the new series is therefore an arbitrary linear function of h successive coefficients of the given series.

If in particular we put

$$\varphi = (1-x)^{h-1}, \quad \text{and} \quad a_n = n^p,$$

where p is a positive integer less than $h-1$, we obtain the closed sum,

$$\sum_h^\infty n^p x^n = \frac{-1}{(1-x)^{h-1}}\sum_{i=1}^{h}\sum_{n=h}^{h+i-2}(-1)^{i-1}\,{}_{h-1}C_{i-1}(n-i+1)^p x^n,$$

where ${}_nC_m$ denotes the coefficient of x^m in $(1+x)^n$.

As a check we calculate

$$4^2x^4 + 5^2x^5 + 6^2x^6 + \cdots = \frac{16x^4 - 23x^5 + 9x^6}{(1-x)^3}, \quad |x| < 1,$$

which may be verified by clearing of fractions.

Evidently (6′) will give in closed form the sum of any convergent power series whose coefficients satisfy a known recurrent relation.

2. The ordinary functions used for developments in series satisfy,

in addition to equation (1), also a relation of the form,

$$\chi_n \psi_n' = \gamma_n \psi_{n+r} + \delta_n \psi_{n+s}, \qquad n = 1, 2, 3, \cdots, \tag{7}$$

where the quantities involved are functions of x defined for each value of n and are continuous for all the values of x in a certain interval. The prime indicates the first derivative with respect to x. We assume that χ_n does not vanish in the given interval, and that r and s are as in theorem 1.

We make use of this relation to obtain a test for the existence of a derivative of the function represented by a given series. In some examples we obtain this derivative in closed form, and an integration then gives the sum of the series.

Let the first k terms of the series be

$$S_k(x) = u_1 \psi_1 + u_2 \psi_2 + \cdots + u_k \psi_k,$$

where $u_1, u_2, \cdots u_k$ are independent of x. Then

$$S_k'(x) = \sum_1^k u_n \psi_n' = \sum_1^k \frac{u_n}{\chi_n} (\gamma_n \psi_{n+r} + \delta_n \psi_{n+s}).$$

By the same process which gave (3), this becomes

$$S_k'(x) = H(x) + G_k(x) + R_k(x),$$

where

$$H(x) = \sum_{n=1}^{r-s} \frac{\delta_n}{\chi_n} u_n \psi_{n+s},$$

$$G_k(x) = \sum_{n=1}^{k} \left(\frac{\gamma_n}{\chi_n} u_n + \frac{\delta_{n+r-s}}{\chi_{n+r-s}} u_{n+r-s} \right) \psi_{n+r}, \tag{8}$$

$$R_k(x) = - \sum_{n=k+1}^{k+r-s} \frac{\delta_n}{\chi_n} u_n \psi_{n+s}.$$

Integrating between limits c and x we have

$$S_k(x) = S_k(c) + \int_c^x H(x)dx + \int_c^x G_k(x)dx + \int_c^x R_k(x)dx.$$

From this follows at once

THEOREM 2. *Let c and x be two points of a closed interval $(c_1 c_2)$, and in this interval assume that*

(α) $\lim_{k=\infty} S_k(c)$ *exists;*

(β) $G_k(x)$ *converges uniformly to a limit* $G(x)$ *when* $k = \infty$;

(γ) $\lim_{k=\infty} \int_c^x R_k(x)dx = 0$, *uniformly.*

Then the series $\sum_1^\infty u_n \psi_n(x)$ *converges uniformly in* (c_1c_2) *to a function* $f(x)$ *whose derivative is*

$$f'(x) = H(x) + G(x).$$

APPLICATIONS.* We shall consider cases in which ψ_n is one of the functions X_n or J_n. Equation (7) becomes respectively

$$(9) \qquad (x^2-1)X'_n = \frac{n(n+1)}{2n+1}(X_{n+1} - X_{n-1});$$

$$(10) \qquad 2J'_n = J_{n-1} - J_{n+1}.$$

In the first case theorem 2 gives the following theorem on the convergence of the series $\sum a_nX_n$, and on the existence of a first derivative of the function represented by this series.

Let c_1 *and* c_2 *be two constants such that* $-1 < c_1 < c_2 < 1$, *and let* c *be a fixed point and* x *a variable point of the interval* (c_1c_2). *Assume that*

(α_1) *the series* $\sum_1^\infty a_nX_n$ *converges when* $x = c$;

(β_1) *the series* $\sum_1^\infty A_nX_n$ *converges uniformly in* (c_1c_2), *where*

$$A_n = \frac{n(n-1)}{2n-1}a_{n-1} - \frac{(n+1)(n+2)}{2n+3}a_{n+1};$$

(γ_1) $\lim_{n=\infty} \frac{a_n}{\sqrt{n}} = 0;$

then the series $\sum_1^\infty a_nX_n$ *converges uniformly in* (c_1c_2) *to a function* $f(x)$ *having a continuous first derivative given by*

$$f'(x) = \frac{-2a_1}{3(x^2-1)} + \frac{1}{x^2-1}\sum_1^\infty A_nX_n.$$

For, comparing (7) and (9) we have

$$\chi_n = (x^2-1); \ \gamma_n = \frac{n(n+1)}{2n+1}; \ \delta_n = -\frac{n(n+1)}{2n+1}; \ r = 1; \ s = -1.$$

The equations (8) become

$$H(x) = \frac{-1}{x^2-1}\left(\tfrac{2}{3}a_1X_0 + \tfrac{6}{5}a_2X_1\right),$$

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* Applications to trigonometric series will be found in the articles by M. Lerch and the writer already cited.

$$G_k(x) = \frac{1}{x^2-1}\sum_1^k\left(\frac{n(n+1)}{2n+1}a_n - \frac{(n+2)(n+3)}{2n+5}a_{n+2}\right)X_{n+1},$$

$$R_k(x) = \frac{1}{x^2-1}\left[\frac{(k+1)(k+2)}{2k+3}a_{k+1}X_k + \frac{(k+2)(k+3)}{2k+5}a_{k+2}X_{k+1}\right].$$

If we note that $X_0 = 1$ and merge the second term of $H(x)$ with $G_k(x)$ we may replace these two expressions respectively by

$$\bar{H}(x) = \frac{-2a_1}{3(x^2-1)},$$

$$\bar{G}_k(x) = \frac{1}{x^2-1}\sum_1^{k+1}\left(\frac{(n-1)n}{2n-1}a_{n-1} - \frac{(n+1)(n+2)}{2n+3}a_{n+1}\right)X_n.$$

By (β_1) the last sum, and hence also $G_k(x)$, approaches a limit uniformly when $k = \infty$. We next show that (γ_1) is a sufficient condition for (γ) of Theorem 2. By means of the identity

$$X_n = \frac{1}{n+1}(X'_{n+1} - xX'_n)$$

the first of the two terms forming $\int_c^x R_k(x)dx$ may be written

$$\frac{k+2}{2k+3}a_{k+1}\int_c \frac{X'_{k+1} - xX'_k}{x^2-1}dx.$$

Separating the integrand into two fractions and integrating each by parts we have, omitting the constant factor,

$$\int_c^x \frac{X'_{k+1}}{x^2-1}dx = \frac{X_{k+1}}{x^2-1}\Big]_c^x + \int_c^x \frac{2xX_{k+1}}{(x^2-1)^2}dx;$$

$$\int_c^x \frac{xX'_k}{x^2-1}dx = \frac{xX_k}{x^2-1}\Big]_c^x + \int_c^x \frac{(x^2+1)X_k}{(x^2-1)^2}dx.$$

On introducing the omitted factor, each term on the right involves either $a_{k+1}X_{k+1}$ or $a_{k+1}X_k$. Under condition (γ_1) both of these products converge uniformly to zero in (c_1c_2). For in this interval we can put X_n in the form*

$$X_n = \frac{\lambda_n + \epsilon_n}{\sqrt{n}},$$

where λ_n and ϵ_n are finite and continuous functions of x with an upper bound independent of n. Hence a constant M exists such that

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*Brand: Les fonctions X_n de Legendre, p. 31. Bonnet: Sur le development des fonctions en series ordonneès suivant les fonctions X et Y.

$$|\sqrt{n}X_n| < M.$$

Therefore

$$|a_nX_n| = \left|\frac{a_n}{\sqrt{n}}\sqrt{n}X_n\right| < M\left|\frac{a_n}{\sqrt{n}}\right|;$$

$$|a_{n+1}X_n| = \left|\frac{a_{n+1}}{\sqrt{n+1}}\sqrt{\frac{n+1}{n}}\sqrt{n}X_n\right| < M\sqrt{\frac{n+1}{n}}\left|\frac{a_{n+1}}{\sqrt{n+1}}\right|.$$

Hence under condition (γ_1),

$$\lim_{n=\infty} a_nX_n = 0 \quad\text{and}\quad \lim_{n=\infty} a_{n+1}X_n = 0.$$

Therefore

$$\lim_{k=\infty}\int_c^x R_k(x)dx = 0$$

uniformly in (c_1c_2), and the theorem follows.

As an example,* consider the series

$$\frac{3}{2}X_1 + \frac{7}{3\cdot 4}X_3 + \frac{11}{5\cdot 6}X_5 + \cdots$$

which converges to zero when $x = 0$. We find

$$f'(x) = \frac{1}{1-x^2} \quad\text{and}\quad f(x) = \tfrac{1}{2}\log\frac{1+x}{1-x},$$

in any interval lying between -1 and $+1$, and not reaching up to one of these points.

Passing to series of Bessel's functions, theorem 2 may be stated as follows:

Let c be a fixed point and x a variable point of any interval (c_1c_2) and assume that

(α_2) *the series* $\sum_1^\infty a_nJ_n$ *converges when* $x = c$;

(β_2) *the series* $\sum_1^\infty (a_{n+2} - a_n)J_{n+1}$ *converges uniformly in* (c_1c_2);

(γ_2) $\lim_{n=\infty} a_{n+1}J_n = 0$ *uniformly in* (c_1c_2).

Then the series $\sum_1^\infty a_nJ_n$ *converges uniformly in* (c_1c_2) *to a value* $f(x)$ *having a continuous first derivative given by*

$$f'(x) = \tfrac{1}{2}(a_1J_0 + a_2J_1) + \tfrac{1}{2}\sum_1^\infty (a_{n+2} - a_n)J_{n+1}.$$

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*Cf. Brand, l. c., p. 99.

Here conditions (α_2), (β_2), (γ_2), are the equivalents of conditions (α), (β), (γ) of Theorem 2 respectively.

As a simple example, let $a_n = 1$, $n = 1, 2, \cdots$. Then we find that the series $\sum_1^\infty J_n(x)$ converges uniformly in any interval to a function $f(x)$ whose derivative is

$$f'(x) = \tfrac{1}{2}(J_0 + J_1).$$

Integrating from 0 to x, since the series converges to 0 when $x = 0$, and noting that

$$\int_0^x J_1 dx = 1 - J_0,$$

we have

$$f(x) = \sum_1^\infty J_n(x) = \tfrac{1}{2} - \tfrac{1}{2}J_0(x) + \tfrac{1}{2}\int_0^x J_0(x)dx.$$

Similarly,

$$\sum_1^\infty (-1)^{n+1}J_n(x) = -\tfrac{1}{2} + \tfrac{1}{2}J_0(x) + \tfrac{1}{2}\int_0^x J_0(x)dx.$$

In closing we indicate another method of utilizing a relation of the form (7), and obtain the sum of the series

$$\sum_1^\infty c^n J_n(x)$$

where c is a constant.

Let

$$S_k(x) = \sum_1^k c^n J_n(x);$$

calculating the expression in (8) we find

$$\begin{aligned} S_k'(x) &= \tfrac{1}{2}\sum_1^k (c^2 - 1)c^n J_{n+1} + \tfrac{1}{2}(cJ_0 + c^2 J_1 - c^{k+1}J_k - c^{k+2}J_{k+1}) \\ &= \frac{c^2-1}{2c} S_k(x) + \tfrac{1}{2}(cJ_0 + J_1 - c^{k+1}J_k - c^k J_{k+1}). \end{aligned}$$

Putting $a = \dfrac{c^2-1}{c}$, and solving the linear differential equation of the first order for $S_k(x)$, we have, noting that $S_k(0) = 0$,

$$S_k(x) = \tfrac{1}{2}e^{ax}\int_0^x (cJ_0 + J_1)e^{-ax}dx - \tfrac{1}{2}e^{ax}\int_0^x c^k(cJ_k + J_{k+1})e^{-ax}dx.$$

If we restrict x by the inequality $|x| < h$, where h is any positive constant, the last integral approaches zero uniformly when $k = \infty$. For, noting that

$$J_k = \frac{x^k}{2^k(k+1)!}\left(1 - \frac{x^2}{2^2(k+1)} + \frac{x^4}{2^4 2!(k+1)(k+2)} - \cdots\right),$$

we can take k so large that

$$|J_k| < \frac{|x^k|}{2^k(k+1)!},$$

or so that

$$|c^kJ_k| < \frac{|c^kx^k|}{2^k(k+1)!}.$$

By comparing the last fraction with the general term of the expansion of $e^{|cx|}$ we see that, for all values of x, $\lim\limits_{k=\infty} c^kJ_k = 0$.

Hence the limiting form of the equation for $S_k(x)$ above is

$$\sum_1^\infty c^nJ_n(x) = \tfrac{1}{2}e^{\frac{c^2-1}{c}x}\int_0^x [cJ_0(x) + J_1(x)]e^{\frac{1-c^2}{c}x}\,dx.$$

When $c = \pm 1$ this reduces to the preceding examples.

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September, 1910.